Gradient of a Scalar Field

$$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$ ← produces a vector

gives the derivative (slope) of T along each direction x, y, z

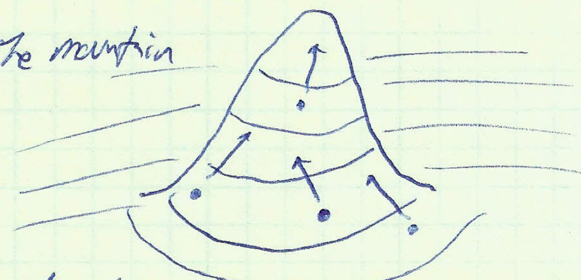
derivative in an arbitrary direction $\hat{a}$ is $\nabla T \cdot \hat{a}$

$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ gradient operator, or "del"

the vector ∇T points in the direction of greatest slope of T

Example: T is a 2D function representing surface of Earth, near mountain

∇T points up the mountain at each point



- In cylindrical coordinates:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

- In spherical coordinates:

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$$

- Note: the extra R and θ terms above are the ~~same~~ inverse of what we had for $d\vec{l}$ in these coordinate systems

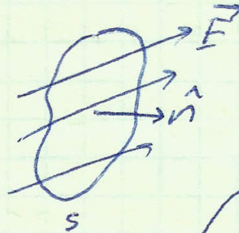
How to remember: In non-cartesian coordinate systems, a unit distance along some directions depends on the value of other coordinates. Think of each $\frac{\partial T}{\partial \square}$ as change in T per unit distance along $\square$

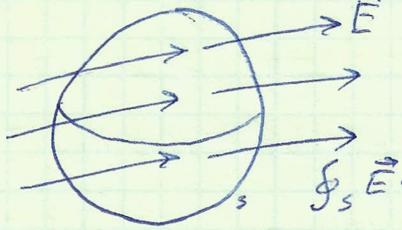
Divergence of a Vector Field

Flux density of field $E = \dfrac{\vec{E}\cdot d\vec{s}}{|d\vec{s}|} = \dfrac{\vec{E}\cdot\hat{n}\,ds}{ds} = \vec{E}\cdot\hat{n}$

Total flux through closed surface

$= \oint_s \vec{E}\cdot d\vec{s}$

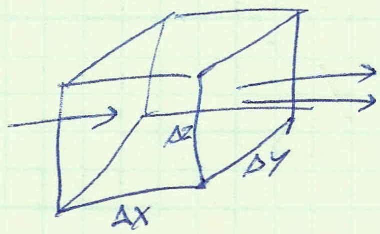




$\oint_s \vec{E}\cdot d\vec{s} = 0$

$\oint_s \vec{E}\cdot d\vec{s} \neq 0$

If a volume contains no sources (charges) then the same amount of flux enters it as leaves it



net flux passing out of volume in x-direction

$$= \underbrace{\frac{\partial E_x}{\partial x}\cdot\Delta x}\cdot\overbrace{\underbrace{\Delta y\cdot\Delta z}_{d\vec{s}}}^{dV}$$

$\vec{E}\cdot\hat{x}$ at second face
$-\vec{E}\cdot\hat{x}$ at first face or ΔE_x

Total flux leaving the volume is (through all six faces)

$$\int_V \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dV$$

$$\boxed{\int_V \nabla\cdot\vec{E}\,dV = \oint_s \vec{E}\cdot d\vec{s}}$$ divergence theorem

produces scalar result from a vector field

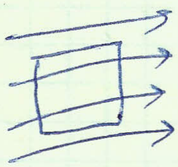
source
positive divergence

sink
negative divergence

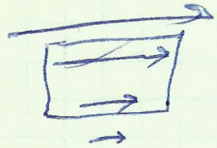
Curl of a Vector Field

Circulation around a closed path

$$\oint_c \vec{B}\cdot d\vec{l}$$



Circulation of uniform field is zero



Field does not necessarily appear "rotational" to have a non-zero circulation.
Example is a field that varies in direction transverse to field direction

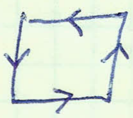
Magnetic field around a wire carrying a current

$$B = \hat{\phi}\frac{\mu_0 I}{2\pi r}$$

$$\text{circulation} = \oint_c \vec{B}\cdot d\vec{l}$$

$$= \int_0^{2\pi} \hat{\phi}\frac{\mu_0 I}{2\pi r}\cdot\hat{\phi}\, r\, d\phi = \mu_0 I$$

Consider a path in the x-y plane



z component of curl is

$$\hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

note that arrows indicate increase in B_y along $\hat{x}$
decrease in B_x along $\hat{y}$

Extend to three dimensions:

$$\nabla\times B = \hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Stoke's theorem $\boxed{\int_s (\nabla\times\vec{B})\cdot d\vec{s} = \oint_c \vec{B}\cdot d\vec{l}}$

Some useful identities

$$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\nabla \times (\nabla V) = 0$$

$$\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E})$$

Laplacian operator

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{(scalar)}$$

$$\nabla^2 \vec{E} = \hat{x} \nabla^2 E_x + \hat{y} \nabla^2 E_y + \hat{z} \nabla^2 E_z \quad \text{(vector)}$$